THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2055 Tutorial 4 (Oct 7)

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(First Three Questions/Solutions)

1. Write down the negations of the following statements.

   (a) $\forall \epsilon > 0$, $\exists N$ such that $\forall n > N$, $|x_n - x| < \epsilon$

   Solution: $\exists \epsilon > 0$, such that $\forall N_0$, $\exists n > N_0$, $|x_n - x| > \epsilon$
   (Typo here: '>' should be changed to '$\geq$'.) □

Explanation of Ng Wing-Kit's Solution

First we introduce a more systematic way of writing down logical propositions.

'bracketed way of writing' [1]

(The following way of rewriting the statement in question 1(a) may clarify what the 'scope' of each 'quantifier' is.)

(Notations: $\mathbb{R}$ = the set of all 'positive real numbers'
$\mathbb{N}$ = the set of all natural numbers, i.e. $0, 1, 2, 3, \cdots$.)

$\forall \epsilon \in \mathbb{R}_+$ ($\exists N \in \mathbb{N}$ ($\forall n \in \mathbb{N}$ ($n > N \implies |x_n - x| < \epsilon$))). $(\star)$

Next, the question 1(a) asks us to negate the statement $(\star)$, which is done in the box below.

Negating the above we get 'by interchanging each '$\forall$' with '$\exists$' the following statement:

$\exists \epsilon \in \mathbb{R}_+$ ($\forall N \in \mathbb{N}$ ($\exists n \in \mathbb{N}$ ($n > N$ and $\sim$ $(|x_n - x| < \epsilon)$))).

$\iff$

$\exists \epsilon \in \mathbb{R}_+$ ($\forall N \in \mathbb{N}$ ($\exists n \in \mathbb{N}$ ($n > N$ and $|x_n - x| \geq \epsilon$))).

Remarks:

- In Ng Wing-Kit's solution, he used the notation N_0 instead of N in order to emphasize the fact that the 'n is greater than 'this' N_0'.
- When we negate expressions like '$\exists\ \epsilon \in \mathbb{R}_+$', we don't negate the phrase '$\in \mathbb{R}_+$', I think this is related to the idea of 'local' versus 'global' variables in programming languages.

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[1] Or you can write the above in a way that resembles computer codes, viz.

Let ϵ denote positive real number; N denote natural number; n denote natural number.

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∀ε (
      ∃N (
            ∀n (
                  n > N ⟹ |x_n − x| < ε ) ) ).
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Remark: In computer languages like visual basic for applications (VBA), statements such as "Let N be nat. no." is written as "Dim N As Integer".

(b) $\exists N$, such that $\forall n > N$, $\forall \epsilon > 0$, $|x_n - x| < \epsilon$

Solution: $\forall N_0$, $\exists n > N_0$, $\exists \epsilon > 0$, $|x_n - x| > \epsilon$ $\square$

In the above line, '$>$' should be changed to '$\geq$'.

(c) $\forall M$, $\exists N$, such that $\forall n > N$, $|x_n - x| > M$

Solution: $\exists M$, $\forall N_0$, $\exists n > N_0$, $|x_n - x| < M$ $\square$

In the above line, '$<$' should be changed to '$\leq$'.

2. For a pair of positive numbers a and b, define sequences a_n and b_n respectively as ('as' should be changed to 'by')

$$a_1 = a, \qquad b_1 = b$$
$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}$$

Prove that $a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ for $n \geq 2$, and (that) they have the same limit.

Solution:
$\forall \alpha, \beta \in \mathbb{R}_+$, (which means the set $(0, \infty)$!)

$$(\sqrt{\alpha} + \sqrt{\beta})^2 \geq 0 \Longrightarrow (\alpha + \beta)/2 \geq \sqrt{\alpha\beta} \tag{1}$$

The first part of the above line should be corrected to $(\sqrt{\alpha} - \sqrt{\beta})^2 \geq 0$, which gives after squaring the following: $\alpha - 2\sqrt{\alpha}\sqrt{\beta} + \beta \geq 0$, which is just inequality (1).

$\therefore \forall n \geq 2$, $a_n \geq b_n$

$b_{n+1} - b_n = \sqrt{a_n b_n} - b_n = \dfrac{b_n(a_n - b_n)}{\sqrt{a_n b_n} + b_n} \geq 0$

Question. Can you figure out why the above line is correct?

$a_n - a_{n+1} = a_n - \left(\dfrac{a_n + b_n}{2}\right) = \dfrac{a_n - b_n}{2} \geq 0$,
therefore starting from $n = 2$, the sequence (a_n) is decreasing and bounded below by b_2.
Also, starting from $n = 2$, the sequence (b_n) is increasing and bounded above by a_2
therefore (a_n) and (b_n) are both convergent sequences.
Next, because (i) $a_{n+1} = \dfrac{a_n + b_n}{2}$, and (ii) (a_n) and (b_n) are both convergent sequences,
we have $\lim\limits_{n\to\infty} a_n = \dfrac{\lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n}{2}$

$\Longrightarrow \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$. $\square$

(The remaining questions as well as solutions will be uploaded next!)